ALGEBRAIC CURVES
EXERCISE SHEET 3

Unless otherwise specified, k is an algebraically closed field.

Exercise 3.1.

(1) Show that $V(Y - X^2) \subset \mathbf{A}^2(\mathbf{C})$ is irreducible; in fact, $I(V(Y - X^2)) = (Y - X^2)$.
(2) Decompose $V(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3) \subset \mathbf{A}^2(\mathbf{C})$ into irreducible components.
(3) Show that $F = Y^2 + X^2(X-1)^2 \in \mathbf{R}[X,Y]$ is an irreducible polynomial, but $V(F)$ is reducible.

Exercise 3.2.

(1) Consider the twisted cubic curve $C = \{(t, t^2, t^3); t \in \mathbf{C}\} \subset \mathbf{A}^3(\mathbf{C})$. Show that C is an irreducible closed subset of $\mathbf{A}^3(\mathbf{C})$. Find generators for the ideal $I(C)$.
(2) Let $V = V(X^2 - YZ, XZ - X) \subset \mathbf{A}^3(\mathbf{C})$. Show that V consists of three irreducible components and determine the corresponding prime ideals.

Exercise 3.3. For topological spaces X and Y, the opens of the product topology on $X \times Y$ are *unions* of products of opens $U \times V$, where $U \subseteq X$ and $V \subseteq Y$. A topological space X is called *Hausdorff* if for any pair of points $x_1 \neq x_2 \in X$, there exist open subsets $U, V \subseteq X$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. A topological space G with an abstract group structure is called a *topological group* if the multiplication and inverse laws are continuous. Let $n \geq 1$.

(1) Is the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$ (each copy of $\mathbb{A}^1_k$ being endowed with the Zariski topology) the same as the Zariski topology on $\mathbb{A}^2_k$?
(2) Is the Zariski topology on $\mathbb{A}^n_k$ Hausdorff?
(3) Is $(\mathbb{A}^n_k, +)$ a topological group for the Zariski topology (assuming $\mathbb{A}^n_k \times \mathbb{A}^n_k \simeq \mathbb{A}^{2n}_k$ is endowed with the Zariski topology)?

Exercise 3.4.

(1) Show that any open subset of an irreducible topological space is irreducible and dense.
(2) Show that the closure of an irreducible subset of a topological space is irreducible.

Exercise 3.5. Let V an affine variety. Show that algebraic subsets of V are in one-to-one correspondence with radical ideals of $\Gamma(V)$. Show that under this correspondence, affine subvarieties correspond to prime ideals and points correspond to maximal ideals.